(a) sequence + limit   (b) series + sum

1) $\dfrac{2^{2n}+3^n}{5^{n-1}} = \dfrac{2^{2n}}{5^{n-1}} + \dfrac{3^n}{5^{n-1}} = \dfrac{4^n}{5^{n-1}} + \dfrac{3^n}{5^{n-1}} \longrightarrow 0.$

$\displaystyle\sum_{n=1}^{\infty}\left(\dfrac{4^n}{5^{n-1}}+\dfrac{3^n}{5^{n-1}}\right) = \sum^{\infty} \dfrac{4^n}{5^{n-1}} \qquad + \qquad \sum^{\infty} \dfrac{3^n}{5^{n-1}}$

geometric,   geometric
ratio $\frac{4}{5}$ ratio $\frac{3}{5}$
1st term 4   1st term 3

$\implies$ sum is $\boxed{\dfrac{4}{1-\frac{4}{5}} + \dfrac{3}{1-\frac{3}{5}}}$ (converges, as both ratios <1).

2) $\dfrac{2^n+3^{2n}}{5^{n+1}} = \underset{\nearrow 0}{\dfrac{2^n}{5^{n+1}}} + \dfrac{9^n}{5^{n+1}} \longrightarrow \infty.$ so sequence diverges.

By nth term test, $\displaystyle\sum \dfrac{2^n+3^{2n}}{5^{n+1}}$ diverges. //

3) $\displaystyle\lim_{n\to\infty} \dfrac{4e^n+n^2}{9e^n+2n} = \dfrac{4e^n+2n}{9e^n+2} = \dfrac{4e^n+2}{9e^n} = \dfrac{4e^n}{9e^n} + \underset{\nearrow 0}{\dfrac{2}{4e^n}} = \dfrac{4}{9}$

         $\uparrow$ $\uparrow$
      L'Hôpital   L'Hôpital

By nth term test, the series diverges.

4) $\dfrac{1}{(n^3+1)^{1/3}} - \dfrac{1}{((n+2)^3+1)^{1/3}} \longrightarrow 0.$

The series telescopes, + starts $\left(\dfrac{1}{2^{1/3}} - \dfrac{1}{(3^3+1)^{1/3}}\right) + \dfrac{1}{(2^3+1)^{1/3}} - \dfrac{1}{(4^3+1)^{1/3}} + \dfrac{1}{(3^3+1)^{1/3}} - \dfrac{1}{(5^3+1)^{1/3}}$

so sum is $\dfrac{1}{2^{1/3}} + \dfrac{1}{(2^3+1)^{1/3}}$ //

5 The sequence a_k is totally unreasonable. But you **can** do them with Taylor series.

$$(n^3+1)^{1/3} = n + \frac{1}{3n^2} + -\frac{1}{9n^5} + \cdots$$

$$((n+2)^3+1)^{1/3} = n+2 + \frac{1}{3n^2} - \frac{4}{3n^3} + \frac{4}{x^4} + \cdots$$

being very fancy w/ Taylor's Theorem (or using what your book calls the generalized binomial theorem).

$$\Rightarrow \lim_{n\to\infty} (n^3+1)^{1/3} - \left((n+2)^3+1\right)^{1/3} = \left(\cancel{n} + \cancel{\frac{1}{3n^2}}^{\;0} + \cancel{\cdots}^{\;0}\right) - \left(\cancel{n} + \underline{2} + \cancel{\frac{1}{3n^2}}^{\;0} + \cancel{\cdots}^{\;0}\right)$$

$$= -2.$$

So the sequence $\longrightarrow -2$.   But I don't expect you to do that. //

The series telescopes, but in the wrong (diverging) way.

(+ it fails n^{th} term test) $\longrightarrow$ diverges. //

6 $\dfrac{2^n+1}{e^n} = \left(\dfrac{2}{e}\right)^n + \left(\dfrac{1}{e}\right)^n \longrightarrow 0$

geometric series $\Rightarrow$ sum is $\dfrac{2/e}{1-2/e} + \dfrac{1/e}{1-1/e}$ ☑

7 $\dfrac{4^n+1}{\pi^n} = \left(\dfrac{4}{\pi}\right)^n + \left(\dfrac{1}{\pi}\right)^n$ $4 > \pi$, so $\left(\dfrac{4}{\pi}\right)^n \longrightarrow \infty$

so sequence diverges + thus the series diverges.

8 $\ln(n(n+1)) = \ln n + \ln n+1$ $\Rightarrow$ $\ln(n(n+1)) - \ln((n+1)^2) = \ln n - \ln(n+1)$

$\ln((n+1)^2) = 2\ln(n+1)$ $= \ln\left(\dfrac{n}{n-1}\right) \longrightarrow 0.$

the series is $\sum \ln n - \ln(n+1)$, which telescopes, but in a divergent way.

So it diverges.